Section 5.3.2, which show that the optimal list price in an inventory-pricing problem (the optimal solution) is decreasing in the inventory on hand (the parameter).

More abstractly, let $X \subset \Re^n$ be a constraint set, $\Theta \subset \Re^l$ be a set of parameter values and $f : X \times \Theta \to \Re$ be an objective function. We need the following definition:

DEFINITION C. 11 *A function $f : X \times \Theta \to \Re$ is said to have increasing differences in $(\mathbf{x}, \boldsymbol{\theta})$, if for all $\mathbf{x}' \geq \mathbf{x}$ and $\boldsymbol{\theta}' \geq \boldsymbol{\theta}$,*

$$f(\mathbf{x}', \boldsymbol{\theta}') - f(\mathbf{x}, \boldsymbol{\theta}') \geq f(\mathbf{x}', \boldsymbol{\theta}) - f(\mathbf{x}, \boldsymbol{\theta}).$$

Define the component-wise minimum (the *meet*) of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\Re^n$ by

$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$$

and the component-wise maximum (the join) of the vectors by

$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}).$$

A set $X \subset \Re^n$ is called a *lattice* if for all $\mathbf{x}$,$\mathbf{y}$ in *X*, the meet and joint of $\mathbf{x}$ and $\mathbf{y}$ are also in *X*. If in addition *X* is compact (closed and bounded), then the set *X* is called a *compact sublattice.* A point $\mathbf{x}^*$ is said to be a *greatest element* (respectively, *least element*) of the sublattice *X* if $\mathbf{x}^* \geq x$ (respectively, $\mathbf{x}^* \leq \mathbf{x}$) for all $\mathbf{x} \in X$. We then have

PROPOSITION C.11 *If X is a nonempty, compact sublattice, then X has a greatest and least element.*

That is, if *X* is a compact sublattice, we can always identify a "largest" and "smallest" (component-wise) element of the set *X*.

Consider next the following definition:

DEFINITION C.12 *A function $f : \Re^n \to \Re$ is said to be supermodular in $\mathbf{z}$ if for all $\mathbf{z}$ and $\mathbf{z}'$ in $\Re^n$,*

$$f(\mathbf{z}) + f(\mathbf{z}') \leq f(\mathbf{z} \wedge \mathbf{z}') + f(\mathbf{z} \vee \mathbf{z}').$$

If f above is a C^2 function, then it is supermodular if and only if the cross-partial derivatives satisfy

$$\frac{\partial^2}{\partial z_i \partial z_j} f(\mathbf{z}) \geq 0, \quad \forall i, j, i \neq j.$$

Sofor C^2 functions, supermodularity corresponds to nonnegativity of the cross-partial derivatives.

Now consider the following optimization problem, for fixed $\boldsymbol{\theta} \in \Theta$,

$$\max \quad f(\mathbf{x}, \boldsymbol{\theta}) \qquad \text{(C.5)}$$
$$\mathbf{x} \in X,$$

and define the *optimal action correspondence* (such as a set of optimal solutions)

$$D^*(\boldsymbol{\theta}) = \{\mathbf{x}^* : f(\mathbf{x}^*, \boldsymbol{\theta}) \geq f(\mathbf{x}, \boldsymbol{\theta}) \;\; \forall \mathbf{x} \in X\}.$$

We want to determine when these optimal solutions are in some sense "increasing" in $\boldsymbol{\theta}$.